NEW TRENDS IN RINGS AND MODULES I
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L-STABLE RINGS

W. K. Nicholson

Department of Mathematics

Abstract. Let L be a function assigning to each ring R a well-deOned set L(R) of left ideals of R: Then an element $a \in R$ is called L-stable when the following condition is satisOed: If $Ra + L = R$; $L \in L(R)$; then a $u \in L$ for some unit $u \in R$: A ring R is called an L-stable ring if each $a \in R$ is L-stable. To rule out uninteresting cases it is assumed that if $\sigma : R \to S$ is a ring isomorphism then the map $L \to \sigma(L)$ is a bijection $L(R) \to L(S)$: When this is the case L is called a left idealtor. A class C of rings is said to be created by a left idealtor L if C equals the class of all L-stable rings. Of course the left idealtor $L(R) = \{L|LisaleftidealofR\}$ creates Bassi rings of stable range 1 (SR1), a fact that motivates this study. However, beside the SR1 rings this theory encompasses two other well known, important classes of rings: (1) The left uniquely generated (UG) rings of Kaplansky: $L(R) = \{l(b)|b \in R\}$; (2) The internal cancellation (IC) rings of Ehrlich: $L(R) = \{Re|e^2 = e \in R\}$: This perspective provides a new tool for the study of these rings, and reveals properties about them not noticed before. This is even more important for the class of directly Onite (DF) rings as it is shown that they too are created by a left idealtor. Relations between various L-stability classes are studied (for example the SR1 rings are clearly L-stable for every L, so all SR1 rings are left UG, IC and DF). Conditions are given on a left idealtor L which guarantee that L-stablity passes to related rings like images, corners, direct products and ideal-extensions (the latter giving examples new even for IC and DF). Vasersteinis theorem that SR1 is a left-right symmetric condition is extended (but this remains open in the important UG case). A left idealtor L is called normal if $L \in L(R)$ implies $u^{-1}Lu \in L(R)$ for any unit u: If L is any normal left idealtor it is shown that the product of L-stable elements is L-stable, a new result even for SR1 rings.

An element $a \in R$ is called unit-regular if aua = a for some unit u: For any left idealtor L it is shown that $a \in R$ is L-stable if a is unit-regular or if $a \in J(R)$- the Jacobson radical. In particular, as for the SR1 rings, all unit-regular rings are L-stable for every L: This makes clear Kaplanskyis observation that every local ring is left UG. The connection between L-stability and unit-regularity is explored and easily shows that R is SR1 if $R/J(R)$ is unit-regular, extending Bassiwell known theorem that every semilocal ring is SR1.

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wknichol@ucalgary.ca.